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The optimal spacing of parallel plates cooled by forced convection

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Abstract—This paper reports the optimal board-to-board spacing and maximum total heat transfer rate from a stack of parallel boards cooled by laminar forced convection. The optimal spacing is proportional to the board length raised to the power 1/2, the property group $(\mu\alpha)^{1/4}$, and $(\Delta P)^{-1/4}$, where ΔP is the pressure head maintained across the stack. The maximum total heat transfer rate is proportional to $(\Delta P)^{1/2}$, the total thickness of the stack (H), and the maximum allowable temperature difference between the board and the coolant inlet. Board surfaces with uniform temperature and uniform heat flux are considered. It is shown that the surface thermal condition (uniform temperature vs uniform heat flux) has a minor effect on the optimal spacing and the maximum total heat transfer rate.

1. INTRODUCTION

THE OBJECTIVE of this study is to determine the optimal spacing for maximum heat transfer from a package (stack) of parallel plates that are cooled by *forced convection*. An application of this arrangement is the forced-air cooling of a stack of electronic circuit boards. Of interest in such an application is the maximum heat transfer, i.e. the maximum density of heat-generating electronics that can be fitted in a package of specified volume.

The optimal board-to-board spacing has been determined for applications in which the cooling is by *natural convection*. This development was communicated in 1984 simultaneously by Bar-Cohen and Rohsenow [1] and Bejan [2]. The most recent reviews of the fundamental heat transfer literature on electronic equipment cooling [3, 4] show that the optimal spacing has not been determined for packages that are cooled by forced convection. The present study fills this void, and develops concrete means for calculating the optimal board-to-board spacing and the associated features of the optimized package.

2. ORDER OF MAGNITUDE ANALYSIS

Consider the geometry of Fig. 1, in which the coolant inlet temperature T_∞ and the pressure head established by the compressor (or pump) ΔP are fixed. In this simple analysis the flow is assumed to be laminar, and the board temperature is assumed to be uniform at the safe level T_w. Each board has a thickness t that is sufficiently smaller than D. To determine the optimal board-to-board spacing D is the same as determining the optimal number of boards ($n \gg 1$) that can fill a space of thickness H

$$n \cong \frac{H}{D}. \tag{1}$$

The following analysis is analogous to the method employed by Bejan [2] to solve the natural convection counterpart of the same design problem.

(a) Consider first the limit $D \to 0$, when each channel is slender enough so that the flow is fully developed all along L. In the same limit, the fluid outlet temperature approaches the board temperature T_w. It is not difficult to show that the average fluid velocity in each channel is

$$U = \frac{D^2}{12\mu}\frac{\Delta P}{L} \tag{2}$$

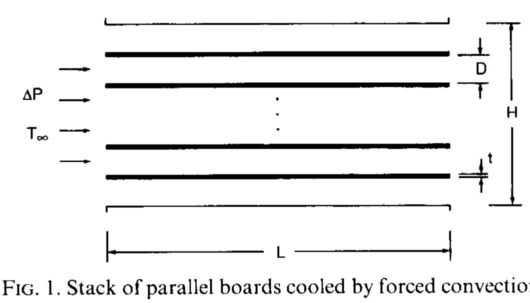

FIG. 1. Stack of parallel boards cooled by forced convection.

NOMENCLATURE

c_P	specific heat at constant pressure
D	board-to-board spacing
D_h	hydraulic diameter, $2D$
f_{app}	apparent friction factor
$\bar{h}$	L-averaged heat transfer coefficient
H	height (thickness) of entire stack, Fig. 1
k	fluid thermal conductivity
L	board length
$\dot{m}'$	mass flow rate per unit length
n	number of channels, H/D
p	dimensionless pressure head, equation (22)
P	pumping power
ΔP	pressure head
Pr	Prandtl number, ν/α
q'	total heat transfer rate per unit length
q'_1	heat transfer rate from one surface per unit length
q''	uniform heat flux
$\bar{q}''$	L-averaged heat flux
q'''	average volumetric heat generation rate, q'/HL
Re	duct Reynolds number, UD_h/ν
Re_L	wall Reynolds number, UL/ν
t	board thickness
T_{out}	coolant outlet temperature
T_w	maximum (allowable) surface temperature
T_∞	coolant inlet temperature
ΔT_{lm}	log-mean temperature difference, equation (18)
U	cross-section averaged longitudinal velocity
U_∞	free stream velocity
x^*	dimensionless longitudinal coordinate for the thermal entrance region, equation (17)
x^+	dimensionless longitudinal coordinate for the hydrodynamic entrance region, equation (20).

Greek symbols

α	fluid thermal diffusivity
δ	dimensionless board-to-board spacing, equation (23)
μ	viscosity
ν	kinematic viscosity, μ/ρ
ρ	fluid density
$\bar{\tau}_w$	L-averaged wall shear stress.

Subscripts

max	maximum
opt	optimal.

and that the total mass flow rate $\dot{m}'$ through the stack of height H is

$$\dot{m}' = \rho UH = \rho H \frac{D^2}{12\mu}\frac{\Delta P}{L}. \quad (3)$$

The mass flow rate $\dot{m}'$ is expressed per unit length in the direction perpendicular to Fig. 1. The total heat transfer rate removed from the entire sandwich by the $\dot{m}'$ stream is

$$q'_a \cong \dot{m}'c_P(T_w - T_\infty) = \rho H \frac{D^2}{12\mu}\frac{\Delta P}{L}c_P(T_w - T_\infty). \quad (4)$$

In conclusion, in the limit $D \to 0$ the total cooling rate (or total rate of allowable Joule heating in the package) decreases as D^2. This trend is illustrated qualitatively as curve (a) in Fig. 2.

(b) In the opposite limit, $D \to \infty$, the boundary layer that lines each surface becomes 'distinct'. In other words, each channel looks like the entrance region to a parallel-plate duct. The overall pressure drop is fixed at ΔP; therefore, the first question here is what is the free-stream velocity U_∞ that sweeps these boundary layers?

The overall force balance on the control volume $H \times L$ requires

$$\Delta P \cdot H = n \cdot 2 \cdot \bar{\tau}_w L \quad (5)$$

in which n is the number of channels and $\bar{\tau}_w$ is the L-averaged wall shear stress

$$\bar{\tau}_w = 1.328 Re_L^{-1/2}\, \tfrac{1}{2}\rho U_\infty^2. \quad (6)$$

Combined, equations (5) and (6) yield

$$U_\infty = \left(\frac{1}{1.328}\frac{\Delta P \cdot H}{nL^{1/2}\rho\nu^{1/2}}\right)^{2/3}. \quad (7)$$

The total heat transfer rate from one of the L-long surface (q'_1) can be calculated by recognizing the overall Nusselt number

$$\frac{\bar{h}L}{k} = \frac{\bar{q}''}{T_w - T_\infty}\frac{L}{k} = 0.664 Pr^{1/3}\left(\frac{U_\infty L}{\nu}\right)^{1/2}. \quad (8)$$

This yields

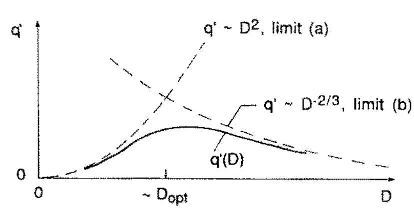

FIG. 2. Determining the optimal spacing by intersecting the asymptotes (4) and (11).

$$q'_1 = \bar{q}''L = k(T_w - T_\infty)0.664 Pr^{1/3}\left(\frac{U_\infty L}{\nu}\right)^{1/2}. \quad (9)$$

The total heat transfer rate released by the entire stack is $2n$ times larger than q'_1 (we are assuming that both surfaces of one board are Joule-heated to T_w):

$$q'_b = 2nq'_1 = 2nk(T_w - T_\infty)0.664 Pr^{1/3}\left(\frac{U_\infty L}{\nu}\right)^{1/2}. \quad (10)$$

In view of the n and U_∞ expressions listed in equations (1) and (7), the total heat transfer rate becomes

$$q'_b = 1.208k(T_w - T_\infty)H\frac{Pr^{1/3}L^{1/3}\Delta P^{1/3}}{\rho^{1/3}\nu^{2/3}D^{2/3}}. \quad (11)$$

The second conclusion we reach is that in the large-D limit the total heat transfer rate decreases as $D^{2/3}$. This second trend has been added as curve (b) to the same graph (Fig. 2), to suggest that the maximum of the actual (unknown) curve $q'(D)$ can only occur at an optimal spacing D_{opt} that is of the same order as the D value obtained by intersecting the asymptotes (4) and (11). It is easy to show that the order of magnitude statement $q'_a \sim q'_b$ yields the optimal spacing for laminar forced convection

$$\frac{D_{opt}}{L} \cong 2.73\left(\frac{\mu\alpha}{\Delta P \cdot L^2}\right)^{1/4} \quad (12)$$

in which μ and α are the viscosity and thermal diffusivity of the coolant.

The order of magnitude of the maximum heat transfer rate that corresponds to $D = D_{opt}$ is obtained by combining equations (4) and (10):

$$q'_{max} \lesssim 0.62\left(\frac{\rho\Delta P}{Pr}\right)^{1/2} Hc_P(T_w - T_\infty). \quad (13)$$

The sign '$\lesssim$' is a reminder that the actual q' maximum is lower than the q' value obtained by intersecting asymptotes (a) and (b) in Fig. 2. The group of properties and dimensions formed on the right side of equation (13) represents the correct scale of q'_{max}. We return to the design implications of this scale in Section 5.

The simple argument presented in this section can be repeated for the situation in which only one surface of the board is Joule-heated to T_w, and the other surface can be modeled as adiabatic. The only change is that $2n$ is replaced by n in equation (10), so that the results (12) and (13) are now replaced by

$$\frac{D_{opt}}{L} \cong 2.10\left(\frac{\mu\alpha}{\Delta P \cdot L^2}\right)^{1/4} \quad (14)$$

$$q'_{max} \lesssim 0.37\left(\frac{\rho\Delta P}{Pr}\right)^{1/2} Hc_P(T_w - T_\infty). \quad (15)$$

It is worth comparing equations (14) and (15) with equations (12) and (13) to see the preservation of the 'correct' scales derived for D_{opt} and q'_{max}. In other words, the change in the thermal boundary conditions of one board-to-board channel affects (by a factor of order 1) only the value of the numerical coefficient in the expressions for D_{opt} and q'_{max}. This observation will be reinforced by a comparison of the results reported in the next two sections.

3. ISOTHERMAL SURFACES

The accuracy of the preceding results can be assessed by considering a more exact analysis of the developing flow and heat transfer in each board-to-board channel. Consider first the case where the two surfaces that form one channel are isothermal (T_w).

The overall Nusselt number for a channel that has a length L comparable with the entrance length is described well by Stephan's [5] empirical formula (see also Shah and London [6])

$$\frac{\bar{q}''D_h}{\Delta T_{lm}k} = 7.55 + \frac{0.024(x^*)^{-1.14}}{1+0.0358(x^*)^{-0.64}Pr^{0.17}}. \quad (16)$$

This formula is valid for $0.1 < Pr < 1000$. In it D_h is the hydraulic diameter (2D), $\bar{q}''$ is the L-averaged heat flux, and

$$x^* = \frac{L/D_h}{UD_h/\alpha} \quad (17)$$

$$\Delta T_{lm} = \frac{(T_w - T_\infty) - (T_w - T_{out})}{\ln[(T_w - T_\infty)/(T_w - T_{out})]}. \quad (18)$$

In equations (17) and (18), U is the cross-section averaged longitudinal velocity and T_{out} is the bulk temperature of the stream in line with the trailing edge of the board (i.e. at the channel outlet).

According to Shah [7] and Shah and London [6], the pressure drop across the channel can be estimated with the formula

$$f_{app}\,Re = \frac{3.44}{(x^+)^{1/2}} + \frac{24 + 0.674/(4x^+) - 3.44/(x^+)^{1/2}}{1 + 0.000029(x^+)^{-2}} \quad (19)$$

in which

$$x^+ = \frac{L/D_h}{UD_h/\nu}, \quad Re = \frac{UD_h}{\nu} \quad (20)$$

$$f_{app} = \frac{D_h}{4L}\frac{\Delta P}{\frac{1}{2}\rho U^2}. \quad (21)$$

The apparent friction factor method (19)–(21) accounts for friction along the two surfaces and the acceleration of the fluid core as the boundary layers thicken along the surfaces.

The D_{opt} scale derived in the preceding section serves as a basis for the definition of the dimensionless pressure drop and board-to-board spacing,

$$p = \frac{\Delta P \cdot L^2}{\mu\alpha} \quad (22)$$

$$\delta = \frac{D_h}{L}\left(\frac{\Delta P \cdot L^2}{\mu\alpha}\right)^{1/4}. \tag{23}$$

Equations (16)–(21) are then rewritten in terms of p, δ and $Re_L = UL/\nu$; however, the resulting expressions are omitted for brevity. We note only that

$$x^+ = \frac{p^{1/2}}{\delta^2 Re_L} = x^* \, Pr. \tag{24}$$

The heat transfer rate extracted from one channel is $2\bar{q}''L$. An additional equation that is needed for the calculation of T_{out} is the first law of thermodynamics for one channel

$$2\bar{q}''L = \rho U D c_P (T_{out} - T_\infty). \tag{25}$$

Finally, the total heat transfer rate extracted from the entire stack can be calculated as $q' = 2n\bar{q}''L$. The resulting expression can be nondimensionalized in the manner anticipated in equation (13),

$$\frac{q'}{(\rho\Delta P/Pr)^{1/2} H c_P (T_w - T_\infty)} = Pr \, Re_L \, p^{-1/2} \frac{T_{out} - T_\infty}{T_w - T_\infty}. \tag{26}$$

The right hand side of this last expression represents the objective function: when maximized, this quantity delivers the magnitude of the leading numerical coefficient that belongs on the right side of the scaling law (13). The results of the maximization procedure are summarized in Table 1. They were obtained parametrically by using equations (16)–(21), (25) and (26). Assumed first was the value of x^+, which played the role of parameter. The numerical work consisted of calculating in order the right side of equation (19), δ, x^*, the group $Re_L \, p^{-1/2}$, the right side of equation (16), the ratio $(T_w - T_\infty)/(T_w - T_{out})$, and finally the right side of equation (26). In this way emerges numerically the one-to-one relationship between the spacing D (or D_h, or δ) and the total heat transfer rate (q').

Figure 3 shows the maximum exhibited by q' with respect to D, for the case $Pr = 0.72$. The optimal dimensionless spacing (3.033 on the abscissa) is only 11% greater than the order-of-magnitude estimate developed in equation (12). Even more impressive is that the use of the rough D value recommended by equation (12) leads to a total heat transfer rate q' that is only 2.5% below the maximum value reported in Table 1.

By reading Table 1 and Fig. 3 we find that even the order-of-magnitude formula for the maximum heat transfer rate (equation (13)) is fairly accurate. That early estimate is between 18 and 29% greater than the q'_{max} value calculated based on the information developed in this section.

Figure 4 illustrates the position of the q' maximum on the x^+ scale when $Pr = 0.72$ (note that in this case, $x^+ \approx x^*$). The last two columns of Table 1 show that the maximum heat transfer rate is achieved when x^* (not x^+) is consistently of order 0.04. This means that the optimal spacing must be such that the board length L is of the same order as the *thermal* entrance length of the parallel-plate channel. This geometric feature agrees with the corresponding result developed for the optimal natural convection cooling of a stack of vertical boards [1, 2].

4. UNIFORM HEAT FLUX SURFACES

When the two surfaces that define one channel release the uniform heat flux q'' the maximum allowable temperature of the board occurs at its trailing edge. In this section T_w denotes that temperature, i.e. the local wall temperature at the downstream distance L.

The local Nusselt number for the entrance region of a constant-flux, parallel-plate channel was reported numerically by Hwang and Fan [8]. It is easy to check that their local Nusselt number data are fitted within 3% by empirical formulas of the Churchill–Usagi [9] type

$$\frac{q''D_h}{(T_w - T_{out})k} = [(0.587 x^{*\,-1/2})^3 + 8.235^3]^{1/3} \quad (Pr = 0.7) \tag{27}$$

$$\frac{q''D_h}{(T_w - T_{out})k} = [(0.359 x^{*\,-1/2})^{1.583} + 8.235^{1.583}]^{1/1.583} \quad (Pr = 10). \tag{28}$$

Equation (25) continues to apply, except that now $\bar{q}''$ is replaced by q''. The equations that in the preceding section described the fluid mechanics of the channel (equations (19)–(21)) remain unchanged.

The results of maximizing the total heat transfer rate numerically are reported in Table 2. These can be compared with the results listed in Table 1 to conclude that the type of thermal boundary condition has little effect on the optimal board-to-board spacing. For the

Table 1. The optimal spacing and maximum total heat transfer rate for a space filled by a stack of parallel boards with both surfaces isothermal

Pr	δ_{opt}	$(D_{opt}/L)(\Delta P \cdot L^2/\mu\alpha)^{1/4}$	$q'_{max}/(\rho\Delta P/Pr)^{1/2} H c_P (T_w - T_\infty)$	x^+	x^*
0.72	6.066	3.033	0.479	0.031	0.0431
6	6.155	3.077	0.522	0.207	0.0346
20	6.156	3.078	0.527	0.675	0.0338
100	6.110	3.055	0.526	3.45	0.0345
1000	6.050	3.025	0.523	35.8	0.0358

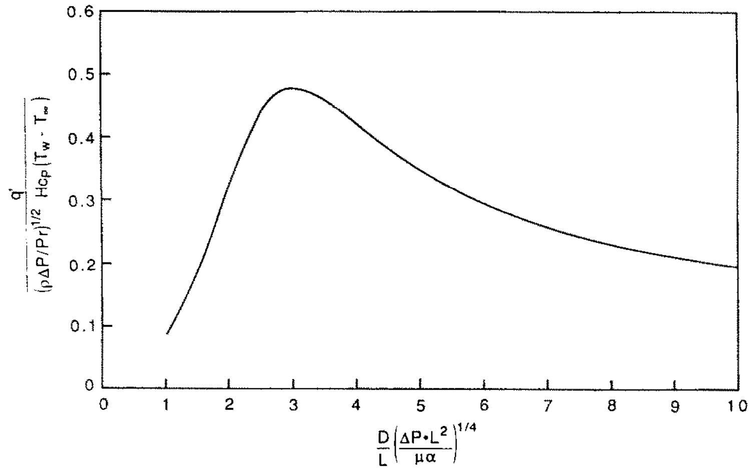

FIG. 3. The optimal board-to-board spacing for maximum heat transfer ($Pr = 0.72$, isothermal surfaces).

same reason, the channel continues to resemble the thermal entrance region between two parallel plates (note the similar x^* values in Table 2 and Table 1).

The maximum total heat transfer rate for the uniform-flux condition is approximately 20% smaller than when the board surfaces are isothermal. This is to be expected because T_w is the ceiling (allowable) surface temperature for both cases, and the isothermal surface has a much higher heat flux near its leading edge than the uniform-flux surface. In other words, the isothermal surface operates at the ceiling level T_w over its entire length, while the uniform-flux surface reaches the ceiling level only at its trailing edge.

5. DISCUSSION

The chief conclusion of this study is that it is possible to select the board-to-board spacing optimally in order to maximize the total heat transfer rate from the $H \times L$ stack to the stream of coolant. This was first demonstrated by means of order-of-magnitude analysis, equations (12) and (13). The same method permitted a quick look into the effect of asymmetric thermal boundary conditions, e.g. a board that generates heat on one side and is insulated on the other side.

The study continued with more exact solutions for the same problem, by assuming isothermal board surfaces (Section 3) and later uniform-flux surfaces (Section 4). The main message of these analyses is that the order-of-magnitude results of Section 2 are correct in a scaling sense and fairly accurate numerically. They also showed that the type of thermal boundary condition has only a minor effect on the optimal board-to-board spacing and the maximum heat transfer rate.

Beginning with equation (12) we learned that the optimal spacing increases as $L^{1/2}$ and decreases as $\Delta P^{-1/4}$. The choice of coolant affects the performance of the assembly through the value of the group of properties $(\mu\alpha)^{1/4}$. For example, if ΔP and L are fixed, the properties are such that the switch from using air (1 atm, 100°C) to using Freon 12 (near-saturated

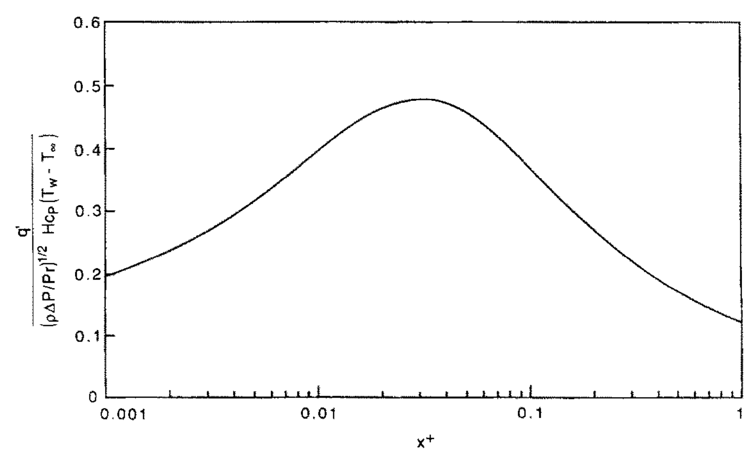

FIG. 4. The position of the heat transfer maximum on the x^+ scale ($Pr = 0.72$, isothermal surfaces).

Table 2. The optimal spacing and maximum total heat transfer rate for a space filled by a stack of parallel boards with uniform heat flux on both surfaces

Pr	δ_{opt}	$(D_{opt}/L)(\Delta P \cdot L^2/\mu\alpha)^{1/4}$	$q'_{max}/(\rho \Delta P/Pr)^{1/2} H c_P (T_w - T_\infty)$	x	x^*
0.7	6.136	3.068	0.371	0.029	0.0416
10	6.574	3.287	0.424	0.264	0.0264

liquid at 367 K) would require a four-fold decrease in the optimal spacing

$$\frac{D_{opt,Freon\ 12}}{D_{opt,air}} = \frac{(\mu\alpha)^{1/4}_{Freon\ 12}}{(\mu\alpha)^{1/4}_{air}} \cong 0.26. \tag{29}$$

The maximum heat transfer rate is proportional to $(T_w - T_\infty)$, H and $(\Delta P)^{1/2}$, equation (13). The corresponding volumetric heat-generating density of the $H \times L$ package increases with $(T_w - T_\infty)$ and $(\Delta P)^{1/2}$, and decreases with L

$$q'''_{max} = \frac{q'_{max}}{HL} \sim (T_w - T_\infty)\frac{(\Delta P)^{1/2}}{L} c_P \left(\frac{\rho}{Pr}\right)^{1/2}. \tag{30}$$

The goodness of the choice of coolant with regard to increasing q'''_{max} is described by the value of the property group $c_P(\rho/Pr)^{1/2}$. Continuing with the numerical example of the preceding paragraph, if $(T_w - T_\infty)$, ΔP and L are fixed, the switch from air to Freon 12 results in a 20-fold increase in the maximum heat generating density,

$$\frac{q'''_{max,Freon\ 12}}{q'''_{max,air}} = \frac{[c_P(\rho/Pr)^{1/2}]_{Freon\ 12}}{[c_P(\rho/Pr)^{1/2}]_{air}} \cong 20. \tag{31}$$

The results of this study are valid when the flow is laminar. This condition acts as a constraint on the pressure difference that is maintained across each channel. The laminar flow condition is the boundary layer criterion $Re_L \lesssim 5 \times 10^5$, because L is of the same order or shorter than the hydrodynamic entrance length when Pr is of order 1 or larger (recall that L is always of the same order as the thermal entrance length). By using equation (24), $\delta \sim 6$ and $x^* \sim 0.04$, the laminar flow criterion reduces to

$$\left(\frac{\Delta P \cdot L^2}{\mu\alpha}\right)^{1/4} \lesssim 10^3\, Pr^{1/2}. \tag{31}$$

Finally, it is worth noting that the q' optimization conclusions reached in this study differ from the conclusions that would be drawn if the $H \times L$ system of Fig. 1 is viewed as a conventional heat exchanger. In the latter, a more meaningful figure of merit is the ratio between the total heat transfer rate, q', and the required pumping power, $\mathbf{P} = \dot{m}'\Delta P/\rho$. It can be shown that this ratio reaches the following asymptotes in the limits (a) and (b) identified in Section 2:

$$\frac{q'_a}{\mathbf{P}} \cong \frac{\rho c_P}{\Delta P}(T_w - T_\infty) \quad (D \to 0) \tag{32}$$

$$\frac{q'_b}{\mathbf{P}} \cong 1.46\frac{\rho c_P}{\Delta P}(T_w - T_\infty)\left(\frac{\mu\alpha}{\Delta P \cdot L^2}\right)^{1/3}\left(\frac{L}{D}\right)^{4/3} Pr^{-1/3} \quad (D \to \infty). \tag{33}$$

These expressions show that the ratio $q'/\mathbf{P}$ is independent of D when $D \to 0$ and decreases as $D^{-4/3}$ as $D \to \infty$. In conclusion, the ratio between total heat transfer rate and pumping power decreases monotonically as the spacing D increases. The ratio $q'/\mathbf{P}$ has the largest value (the constant of equation (32)) in the limit $D \to 0$.

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